

DYNAMICS IN A NONCOMMUTATIVE PHASE SPACE

R.P.Malik,^{* †} A. K. Mishra[‡] and G. Rajasekaran[§]*Institute of Mathematical Sciences
C.I.T. Campus, Madras - 600 113, India*

ABSTRACT

Dynamics has been generalized to a noncommutative phase space. The noncommuting phase space is taken to be invariant under the quantum group $GL_{q,p}(2)$. The q -deformed differential calculus on the phase space is formulated and using this, both the Hamiltonian and Lagrangian forms of dynamics have been constructed. In contrast to earlier forms of q -dynamics, our formalism has the advantage of preserving the conventional symmetries such as rotational or Lorentz invariance.

1 Introduction

Quantum groups, q -deformations and the associated noncommutative quantum plane have been a subject of considerable interest in mathematical physics during the past few years [1–17]. However, despite many attempts [18–25], the key concepts of quantum groups have not yet found a firm footing in the domain of physical applications.

Recently Lukin, Stern and Yakushin [19] developed a dynamical formalism on a 2D quantum plane by exploiting the notion of a tangent space defined over a 2D q -deformed configuration space. In this approach, a differential calculus, that is invariant under the quantum group $GL_{q,p}(2)$, is developed and this is then applied to the construction of dynamics. The conventional symmetries such as rotational invariance and Lorentz invariance are however lost in this dynamics. Furthermore, the status of a single dimensional system is not clear. In an altogether different approach, Aref'eva and Volovich [18] introduced

^{*}On leave of absence from Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna, Moscow, Russia.

[†] e-mail address: malik@thsun1.jinr.dubna.su

[‡]e-mail address: mishra@imsc.ernet.in

[§] e-mail address: graj@imsc.ernet.in

the deformation parameter in the phase space for the description of the free nonrelativistic particle and the harmonic oscillator which was then extended to the multidimensional case [20]. However, a consistent differential calculus in the phase space was not developed in these works, and as a consequence, some of their results are of an adhoc nature.

In the present work, we shall take the phase space as a 2D quantum plane based on the quantum group $GL_{q,p}(2)$ and try to construct dynamics of one-dimensional as well as multidimensional systems, based on a consistent differential calculus. We shall adapt the 2D differential calculus of Lukin et al. for our purpose. We are thus able to complete the program initiated in Refs. [18, 20] by using the mathematical tools invented by Lukin et al. [19]. The resulting dynamics, although q -deformed, preserves conventional symmetries such as rotational and Lorentz invariance.

The material of our work is organised as follows. In Sec. 2, we develop a $GL_{q,p}(2)$ invariant differential calculus in the 2D phase space of a one dimensional system and construct dynamics on the q -deformed symplectic manifold. We also furnish examples of dynamics for a general class of Lagrangians expressed as a polynomial of second degree in position and velocity. We develop a differential calculus in the multidimensional phase space in Sec. 3 and show that the mutual consistency of the $GL_{q,p}(2)$ invariance of the phase space with the separate $GL(N)$ invariance of the configuration and momentum spaces requires the parameters to satisfy $pq = 1$. We then construct the multidimensional dynamics and discuss examples including a free massless relativistic particle. Finally, Sec. 4 is devoted to summary and discussion.

2 Single dimensional systems

2.1 Differential calculus

We construct here a $GL_{q,p}(2)$ -invariant differential calculus for a dynamical system on a 2D cotangent manifold (i.e., phase space) corresponding to a one-dimensional configuration manifold. The phase space variables, the coordinate x and momentum π , are functions of a commuting evolution parameter t in terms of which the trajectory of the physical system is parametrized in the phase space. We introduce a variational derivative δ which is identified with the exterior derivative of the differential geometry d (i.e., $d^2 = \delta^2 = 0$). Our starting point is the following set of basic relations among $x, \pi, \delta x$ and $\delta \pi$:

$$\begin{aligned} x \pi &= q \pi x, \\ x \delta x &= pq \delta x x, \\ x \delta \pi &= q \delta \pi x + (pq - 1) \delta x \pi, \\ \pi \delta \pi &= pq \delta \pi \pi, \\ \pi \delta x &= p \delta x \pi, \end{aligned} \tag{2.1}$$

where q, p are nonzero c-numbers which may be complex.

The algebra given by (2.1) is invariant under the following $GL_{q,p}(2)$ transformation

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad (2.2)$$

where the pair (ϕ, ψ) stands for (x, π) or $(\delta x, \delta \pi)$ which commutes with the elements A, B, C and D of a 2×2 $GL_{q,p}(2)$ matrix obeying the braiding relations in rows and columns as given below

$$\begin{aligned} A B &= p B A, & A C &= q C A, & B C &= (q/p) C B, & B D &= q D B, \\ C D &= p D C, & AD - DA &= (p - q^{-1})BC = (q - p^{-1})CB. \end{aligned} \quad (2.3)$$

We shall use the definition of the partial derivative

$$\delta F = \delta x \left(\frac{\partial F}{\partial x} \right) + \delta \pi \left(\frac{\partial F}{\partial \pi} \right), \quad (2.4)$$

and the Leibnitz rule [11]

$$\delta(FG) = (\delta F) G + F (\delta G), \quad (2.5)$$

where F and G are polynomial functions of x and π .

We next define $\delta x = \dot{x} \delta t$ and $\delta \pi = \dot{\pi} \delta t$ and get from (2.1), the relations

$$\begin{aligned} x \dot{x} &= pq \dot{x} x, \\ x \dot{\pi} &= q \dot{\pi} x + (pq - 1) \dot{x} \pi, \\ \pi \dot{\pi} &= pq \dot{\pi} \pi, \\ \pi \dot{x} &= p \dot{x} \pi. \end{aligned} \quad (2.6)$$

Following Ref. [19], we also postulate

$$\dot{x} \dot{\pi} = q \dot{\pi} \dot{x}. \quad (2.7)$$

All qp -relations with a single variation and single “time” derivative can be derived from (2.1) and (2.6) by exploiting Leibnitz rule (2.5). These are listed below

$$\begin{aligned} \dot{x} \delta x &= \delta x \dot{x}, \\ \dot{\pi} \delta \pi &= \delta \pi \dot{\pi}, \\ \dot{x} \delta \pi &= q \delta \pi \dot{x}, \\ x \delta \dot{x} &= pq \delta \dot{x} x + (pq - 1) \delta x \dot{x}, \\ \pi \delta \dot{\pi} &= pq \delta \dot{\pi} \pi + (pq - 1) \delta \pi \dot{\pi}, \\ x \delta \dot{\pi} &= q \delta \dot{\pi} x + (pq - 1) (\delta \dot{x} \pi + \delta x \dot{\pi}), \\ \pi \delta \dot{x} &= p \delta \dot{x} \pi + (pq - 1) \delta \pi \dot{x}, \\ \dot{\pi} \delta x &= p \delta x \dot{\pi} - (pq - 1) \delta \pi \dot{x}. \end{aligned} \quad (2.8)$$

The relations involving “time” derivatives and their variations are

$$\begin{aligned}
\dot{x} \delta \dot{x} &= qp \delta \dot{x} \dot{x}, \\
\dot{\pi} \delta \dot{\pi} &= qp \delta \dot{\pi} \dot{\pi}, \\
\dot{x} \delta \dot{\pi} &= q \delta \dot{\pi} \dot{x} + (pq - 1) \delta \dot{x} \dot{\pi}, \\
\dot{\pi} \delta \dot{x} &= p \delta \dot{x} \dot{\pi}.
\end{aligned} \tag{2.9}$$

The following wedge products complete the full algebraic structure of the differential calculus

$$\begin{aligned}
\delta x \wedge \delta \dot{x} &= -\delta \dot{x} \wedge \delta x, \\
\delta \pi \wedge \delta \dot{\pi} &= -\delta \dot{\pi} \wedge \delta \pi, \\
\delta x \wedge \delta \pi &= -p^{-1} \delta \pi \wedge \delta x, \\
\delta \dot{x} \wedge \delta \dot{\pi} &= -p^{-1} \delta \dot{\pi} \wedge \delta \dot{x}, \\
\delta \pi \wedge \delta \dot{x} &= -q^{-1} \delta \dot{x} \wedge \delta \pi, \\
\delta x \wedge \delta \dot{\pi} &= -p^{-1} \delta \dot{\pi} \wedge \delta x + (pq)^{-1} (1 - pq) \delta \dot{x} \wedge \delta \pi.
\end{aligned} \tag{2.10}$$

All the qp -algebraic relations (2.1) , (2.6–2.10) can be shown to be invariant under the $GL_{q,p}(2)$ transformations (2.2) if the pair (ϕ, ψ) are taken to be (x, π) , $(\delta x, \delta \pi)$, $(\dot{x}, \dot{\pi})$ or $(\delta \dot{x}, \delta \dot{\pi})$. Furthermore, it can be checked that when any of these quantities ϕ or ψ is commuted through the above algebraic relations, no new secondary relations emerge. This establishes the associativity conditions which are equivalent to the validity of the Yang-Baxter equations.

The construction of the algebraic structure of the qp -deformed differential calculus given in (2.1) and (2.6–2.10) parallels the work of Lukin et al. [19]. Whereas Lukin et al. apply it to the two dimensional configuration space, we apply it to the two dimensional phase space.

2.2 Dynamics

We begin with the action S which is expressed in terms of the Hamiltonian function $H(x, \pi)$ as

$$S = \int dt [\pi \dot{x} - H(x, \pi)]. \tag{2.11}$$

Using (2.4) and (2.5), the action principle can be written as

$$\delta S = 0 = \int dt \left[\delta \pi \dot{x} + \pi \delta \dot{x} - \delta x \frac{\partial H}{\partial x} - \delta \pi \frac{\partial H}{\partial \pi} \right]. \tag{2.12}$$

The following Hamilton equations of motion emerge

$$\begin{aligned}
\dot{x} &= \frac{1}{pq} \frac{\partial H}{\partial \pi}, \\
\dot{\pi} &= -\frac{1}{p} \frac{\partial H}{\partial x},
\end{aligned} \tag{2.13}$$

when we use

$$\pi \delta \dot{x} = p \delta \dot{x} \pi + (pq - 1) \delta \pi \dot{x},$$

from (2.8) while taking all the variations to the left. If we now define the qp -deformed Poisson bracket between two dynamical variables $F(x, \pi)$ and $G(x, \pi)$ as

$$\{F, G\} = \frac{1}{qp} \frac{\partial G}{\partial \pi} \frac{\partial F}{\partial x} - \frac{1}{p} \frac{\partial G}{\partial x} \frac{\partial F}{\partial \pi}, \quad (2.14)$$

the equations of motion (2.13) become

$$\begin{aligned} \dot{x} &= \{x, H\}, \\ \dot{\pi} &= \{\pi, H\}, \end{aligned} \quad (2.15)$$

and the basic canonical brackets turn out to be

$$\begin{aligned} \{x, \pi\} &= (pq)^{-1}, \\ \{\pi, x\} &= -p^{-1}, \\ \{x, x\} &= \{\pi, \pi\} = 0. \end{aligned} \quad (2.16)$$

It may be noted that the above basic brackets remain invariant under the $SL_q(2)$ transformations defined by (2.2) and (2.3) with $q = p$ and the corresponding q -determinant is chosen to be unity, i.e.;

$$A D - q B C = D A - q^{-1} C B = 1.$$

We now turn to the Lagrangian dynamics. We obtain the Lagrangian function $L(x, \dot{x})$ by the Legendre transformation

$$L = \pi \dot{x} - H(x, \pi),$$

where, on the r.h.s., the canonical momentum (π) is to be eliminated using

$$\pi = \frac{1}{p} \frac{\partial L}{\partial \dot{x}}. \quad (2.17)$$

Rewriting the action $S = \int dt L(x, \dot{x})$ and requiring

$$\delta S = 0 \equiv \int dt \left(\delta x \frac{\partial L}{\partial x} + \delta \dot{x} \frac{\partial L}{\partial \dot{x}} \right), \quad (2.18)$$

we obtain the Euler-Lagrange equations of motion

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right), \quad (2.19)$$

which is consistent with the Hamilton equations (2.13) if we take into account the definition of momentum (2.17).

An important feature of the q -deformed dynamics is the existence of certain restrictions on the form of the Lagrangian. These follow from the qp -commutation relations:

$$\begin{aligned} x \pi &= q \pi x, & \dot{x} \dot{\pi} &= q \dot{\pi} \dot{x}, & \pi \dot{\pi} &= pq \dot{\pi} \pi, \\ \pi \dot{x} &= p \dot{x} \pi, & x \dot{\pi} &= q \dot{\pi} x + (pq - 1) \dot{x} \pi, \end{aligned}$$

and these restrictions are

$$\begin{aligned}
x \frac{\partial L}{\partial \dot{x}} &= q \frac{\partial L}{\partial \dot{x}} x, & \dot{x} \frac{\partial L}{\partial x} &= q \frac{\partial L}{\partial x} \dot{x}, \\
\frac{\partial L}{\partial \dot{x}} \frac{\partial L}{\partial x} &= pq \frac{\partial L}{\partial x} \frac{\partial L}{\partial \dot{x}}, & \frac{\partial L}{\partial \dot{x}} \dot{x} &= p \dot{x} \frac{\partial L}{\partial \dot{x}}, \\
x \frac{\partial L}{\partial x} &= q \frac{\partial L}{\partial x} x + (pq - 1) \dot{x} \frac{\partial L}{\partial \dot{x}}.
\end{aligned} \tag{2.20}$$

Similar restrictions can be written for the Hamiltonian too.

More generally we may start with the following action defined on a qp -deformed symplectic manifold

$$S = \int dt \left[\frac{1}{1 + pq} \sum_{M,N} \Omega_{MN} z^M \dot{z}^N - H(z) \right], \tag{2.21}$$

where $z^M = (x, \pi)$ for $M = 1, 2$ is an arbitrary phase space coordinate on the symplectic manifold and Ω_{MN} is a covariant symplectic metric that satisfies

$$\begin{aligned}
\sum_N \Omega_{MN} \Omega^{NL} &= \delta_M^L = \sum_N \Omega^{LN} \Omega_{NM}, \\
\Omega_{MN} &= -q \Omega_{NM}, \quad M \geq N.
\end{aligned} \tag{2.22}$$

Here Ω^{MN} is the contravariant metric and the second condition is the q -antisymmetry property of the covariant symplectic metric. The action principle with the above action (2.21) leads to the Hamilton equations:

$$\dot{z}^M = \sum_N \Omega^{MN} \frac{\partial H}{\partial z^N} \equiv \{z^M, H\}, \tag{2.23}$$

where the general qp -Poisson bracket is defined as [21,26]

$$\{F, G\} = \sum_{M,N} \Omega^{MN} \partial_N G \partial_M F, \tag{2.24}$$

with $\partial_M = \frac{\partial}{\partial z^M}$. A general form of the symplectic metric that satisfies (2.22) is

$$\begin{aligned}
\Omega^{MN} &= \begin{pmatrix} 0, & (hq)^{-1} \\ -h^{-1}, & 0 \end{pmatrix}, \\
\Omega_{MN} &= \begin{pmatrix} 0, & -h \\ hq, & 0 \end{pmatrix},
\end{aligned} \tag{2.25}$$

where h is an arbitrary function of q and p . The dynamics formulated in equations (2.12)–(2.16) is obtained if we choose $h = p$.

2.3 Examples

We begin with the general form for the Lagrangian as a polynomial of the second degree in velocity (\dot{x}) and position (x), namely;

$$L(x, \dot{x}) = \sum_{n,m \geq 0}^2 a_{nm} \dot{x}^n x^m, \quad (2.26)$$

where a_{nm} are the time-independent noncommuting parameters. Since the terms a_{00} , $a_{10}x$ and $a_{11}\dot{x}x$ do not appear in the equation of motion, we shall put $a_{00} = a_{10} = a_{11} = 0$. The restrictions on the Lagrangian given by (2.20) lead to rather complicated restrictions on a_{nm} , in general. We first consider the simple case when $pq = 1$. Then, all the restrictions (2.20) are satisfied if we require the following q -commutation relations between a_{nm} and ξ ($= x, \dot{x}, \delta x, \delta \dot{x}$)

$$\xi a_{nm} = q a_{nm} \xi, \quad (2.27)$$

and following amongst a_{nm} :

$$\begin{aligned} a_{02}a_{20} &= a_{20}a_{02}, & a_{12}a_{21} &= q a_{21}a_{12}, & a_{01}a_{02} &= q a_{02}a_{01}, & a_{01}a_{20} &= q a_{20}a_{01}, \\ a_{01}a_{12} &= q^2 a_{12}a_{01}, & a_{01}a_{21} &= q^2 a_{21}a_{01}, & a_{01}a_{22} &= q^3 a_{22}a_{01}, & a_{02}a_{12} &= q a_{12}a_{02}, \\ a_{02}a_{21} &= q a_{21}a_{02}, & a_{20}a_{12} &= q a_{12}a_{20}, & a_{20}a_{21} &= q a_{21}a_{20}, \\ a_{20}a_{22} &= q^2 a_{22}a_{20}, & a_{12}a_{22} &= q a_{22}a_{12}, & a_{21}a_{22} &= q a_{22}a_{21}. \end{aligned} \quad (2.28)$$

We now find the dynamical equation of motion

$$(a_{20} + a_{21} x + a_{22} x^2) \ddot{x} + \frac{1}{2} (a_{21} + 2 a_{22} x) \dot{x}^2 - a_{02} x - \frac{1}{2} a_{01} = 0. \quad (2.29)$$

The cononical momentum π and the Hamiltonian H are

$$\pi = q \left(\frac{\partial L}{\partial \dot{x}} \right) \equiv 2 (a_{20} + a_{21} x + a_{22} x^2) \dot{x} + a_{12} x^2, \quad (2.30)$$

$$H = \frac{1}{4} (\pi - a_{12} x^2) (a_{20} + a_{21} x + a_{22} x^2)^{-1} (\pi - a_{12} x^2) - a_{01} x - a_{02} x^2. \quad (2.31)$$

The Hamilton equations of motion are:

$$\dot{\pi} = -q \frac{\partial H}{\partial x} = a_{01} + 2a_{02}x + 2a_{12}\dot{x}x + 2a_{22}\dot{x}^2x + a_{21}\dot{x}^2, \quad (2.32)$$

$$\dot{x} = \frac{\partial H}{\partial \pi} \equiv \frac{1}{2} (a_{20} + a_{21} x + a_{22} x^2)^{-1} (\pi - a_{12} x^2), \quad (2.33)$$

and these are consistent with (2.29).

For further analysis, we specialise to the class of systems with $a_{12} = a_{21} = a_{22} = 0$. The Lagrangian, the Hamiltonian, the canonical momentum and the equation of motion are

$$L = a_{20} \dot{x}^2 + a_{02} x^2 + a_{01} x, \quad (2.34)$$

$$H = \frac{1}{4} \pi a_{20}^{-1} \pi - a_{01} x - a_{02} x^2, \quad (2.35)$$

$$\pi = 2 a_{20} \dot{x}, \quad (2.36)$$

$$\ddot{x} = a_{20}^{-1} a_{02} x + \frac{1}{2} a_{20}^{-1} a_{01} \equiv \omega^2 x + C, \quad (2.37)$$

where

$$\omega^2 = a_{20}^{-1} a_{02} \quad (2.38)$$

$$C = \frac{1}{2} a_{20}^{-1} a_{01}. \quad (2.39)$$

The solution is

$$x(t) = e^{\omega t} A + e^{-\omega t} B - \omega^{-2} C, \quad (2.40)$$

where A and B are constants which may be noncommuting in general and they can be determined in terms of $x(0)$ and $\dot{x}(0)$:

$$A = \frac{1}{2} \left[x(0) + \omega^{-2} C + \omega^{-1} \dot{x}(0) \right], \quad (2.41)$$

$$B = \frac{1}{2} \left[x(0) + \omega^{-2} C - \omega^{-1} \dot{x}(0) \right]. \quad (2.42)$$

Note that (2.38) defines only ω^2 , but we assume that ω and ω^{-1} also exist. The basic q -commutation relation from (2.6) at $t = 0$ is

$$x(0) \dot{x}(0) = \dot{x}(0) x(0). \quad (2.43)$$

Using (2.43), (2.27) and (2.28), we find

$$A B = B A, \quad B C = C B, \quad C A = A C, \quad (2.44)$$

$$\omega A = A \omega, \quad \omega B = B \omega, \quad \omega C = C \omega, \quad (2.45)$$

$$\omega x(0) = x(0) \omega, \quad \omega \dot{x}(0) = \dot{x}(0) \omega, \quad (2.46)$$

Further, it can be verified that all the q -commutation relations among x, π, \dot{x} and $\dot{\pi}$ are satisfied for all values of t . *Thus, we have a completely consistent dynamics in a non-commutative phase space, for the system defined by either (2.34) or (2.35).* However, this consistency has been obtained at a price; all the constants A, B, C and ω that determine the solution $x(t)$ in (2.40), commute with each other.

Let us again consider the Lagrangian of (2.34) and ask whether a more general dynamics with $pq \neq 1$ can be constructed from it. We find that the restrictions (2.20) can be satisfied for $pq \neq 1$ if we take the q -commutation relations of a_{nm} with $\xi(=x, \dot{x}, \delta x, \delta \dot{x})$:

$$\xi a_{01} = q a_{01} \xi, \quad \xi a_{02} = p q^2 a_{02} \xi, \quad \xi a_{20} = p^{-1} a_{20} \xi, \quad (2.47)$$

and q -commutation relations among a_{nm}

$$a_{20} a_{01} = p a_{01} a_{20}, \quad a_{20} a_{02} = a_{02} a_{20}, \quad (2.48)$$

but, we also require

$$(pq - 1) (a_{02} x^2 - pq a_{20} \dot{x}^2) = 0, \quad (2.49)$$

which follows from the relation

$$x \dot{\pi} = q \dot{\pi} x + (pq - 1) \dot{x} \pi.$$

As a consequence of the restriction (2.49), dynamics for $pq \neq 1$ does not evolve in the two-dimensional phase space, but appears to degenerate into a restricted one-dimensional region defined by (2.49).

Further study of the time evolution of the system confirms the above conclusion. The equation of motion

$$\ddot{x} = \frac{1}{p^2 q^2} a_{20}^{-1} a_{02} x + \frac{1}{pq(1+pq)} a_{20}^{-1} a_{01}, \quad (2.50)$$

has the solution:

$$x(t) = e^{\omega t} A + e^{-\omega t} B - \omega^{-2} C, \quad (2.51)$$

where

$$\omega^2 = (pq)^{-2} a_{20}^{-1} a_{02}, \quad (2.52)$$

$$C = (pq)^{-1} (1 + pq)^{-1} a_{20}^{-1} a_{01}, \quad (2.53)$$

and A and B can be again written as:

$$A = \frac{1}{2} \left[x(0) + \omega^{-2} C + \omega^{-1} \dot{x}(0) \right], \quad (2.54)$$

$$B = \frac{1}{2} \left[x(0) + \omega^{-2} C - \omega^{-1} \dot{x}(0) \right]. \quad (2.55)$$

To satisfy (2.49) at $t = 0$, we put $C = 0$ and so we drop the a_{01} term in the Lagrangian but, in addition, we require either $B = 0$ or $A = 0$. As a consequence, we have

$$\dot{x}(0) = \pm \omega x(0). \quad (2.56)$$

The corresponding solutions are therefore either

$$x(t) = e^{\omega t} A \quad \text{with} \quad A \omega = pq \omega A, \quad (2.57)$$

or,

$$x(t) = e^{-\omega t} B \quad \text{with} \quad B \omega = pq \omega B. \quad (2.58)$$

One can verify that either of these solutions satisfies all the consistency conditions including the restriction (2.49) ($a_{02}x^2 = pq a_{20}\dot{x}^2$) for all t . Because of (2.56), these solutions do not allow arbitrary and independent initial values for $x(0)$ and $\dot{x}(0)$. In other words, the phase space point at $t = 0$ in the two-dimensional phase space must be chosen to lie in the restricted “one dimensional phase space” and the subsequent dynamics evolves within this one-dimensional space. Note that in contrast to the case of $pq = 1$, the two constants ω and A (or B) on which $x(t)$ depends, do not commute with each-other, still there exists a consistent time evolution, albeit a restricted one.

Before we close this subsection, we remark that the examples discussed here, not only generalize the “ q -deformed free particle and harmonic oscillator systems” studied in Refs. [20] and [18], but also incorporate them in a well-defined framework of q -deformed differential calculus. Thus, our formalism provides a sound basis for the heuristic results derived earlier in the literature.

3 Multidimensional systems

3.1 Differential calculus

We begin with an N dimensional configuration space which is undeformed (i.e., $x_i x_j = x_j x_i, i, j = 1, 2, 3, \dots, N$). In the corresponding $2N$ dimensional momentum phase space (cotangent manifold), we introduce the q -deformation in such a way as to preserve the $GL(N)$ invariance in the configuration as well as the momentum space separately. We take for all i and j

$$x_i x_j = x_j x_i, \quad \pi_i \pi_j = \pi_j \pi_i, \quad x_i \pi_j = q \pi_j x_i, \quad (3.1)$$

where π_i are the conjugate momenta corresponding to the coordinates x_i of the configuration space and q is a nonzero c-number. The above set of relations are invariant under the $GL(N)$ transformations

$$x'_i = a_{ij} x_j, \quad \pi'_i = a_{ij} \pi_j, \quad (3.2)$$

where a_{ij} are the commuting c-number elements of a $N \times N$ nonsingular matrix of the undeformed general linear group of transformations $GL(N)$. If we now apply the quantum group transformations

$$\begin{aligned} x_i &\rightarrow A x_i + B \pi_i, \\ \pi_i &\rightarrow C x_i + D \pi_i, \end{aligned} \quad (3.3)$$

where A, B, C, D are the elements of a 2×2 $GL_{q,p}(2)$ matrix obeying the relations given in (2.3), the q -algebraic relations (3.1) remain form invariant only if $pq = 1$. Here the commutativity of the group elements with the dynamical variables is assumed and the transformations (3.3) are applied between pairs of conjugate variables: $(x_1, \pi_1), (x_2, \pi_2), \dots, (x_N, \pi_N)$. Under the restriction $pq = 1$, the relations (2.3) reduce to a simpler form:

$$\begin{aligned} A B &= q^{-1} B A, & A C &= q C A, & B C &= q^2 C B, \\ C D &= q^{-1} D C, & A D &= D A, & B D &= q D B. \end{aligned} \quad (3.4)$$

We now develop the differential calculus that is invariant under (3.2) and (3.3). The

differential calculus is based on the following relations in addition to (3.1):

$$\begin{aligned}
x_i \dot{x}_j &= \dot{x}_j x_i, & \pi_i \dot{\pi}_j &= \dot{\pi}_j \pi_i, & x_i \dot{\pi}_j &= q \dot{\pi}_j x_i, & \pi_i \dot{x}_j &= q^{-1} \dot{x}_j \pi_i, \\
x_i \delta x_j &= \delta x_j x_i, & \pi_i \delta \pi_j &= \delta \pi_j \pi_i, & x_i \delta \pi_j &= q \delta \pi_j x_i, & \pi_i \delta x_j &= q^{-1} \delta x_j \pi_i, \\
\dot{x}_i \delta x_j &= \delta x_j \dot{x}_i, & \dot{\pi}_i \delta \pi_j &= \delta \pi_j \dot{\pi}_i, & x_i \delta \dot{x}_j &= \delta \dot{x}_j x_i, & \pi_i \delta \dot{\pi}_j &= \delta \dot{\pi}_j \pi_i, \\
\dot{x}_i \delta \pi_j &= q \delta \pi_j \dot{x}_i, & \dot{\pi}_i \delta x_j &= q^{-1} \delta x_j \dot{\pi}_i, & x_i \delta \dot{\pi}_j &= q \delta \dot{\pi}_j x_i, & \pi_i \delta \dot{x}_j &= q^{-1} \delta \dot{x}_j \pi_i, \\
\dot{x}_i \dot{x}_j &= \dot{x}_j \dot{x}_i, & \dot{\pi}_i \dot{\pi}_j &= \dot{\pi}_j \dot{\pi}_i, & \dot{x}_i \delta \dot{x}_j &= \delta \dot{x}_j \dot{x}_i, & \dot{\pi}_i \delta \dot{\pi}_j &= \delta \dot{\pi}_j \dot{\pi}_i, \\
\dot{x}_i \dot{\pi}_j &= q \dot{\pi}_j \dot{x}_i, & \dot{\pi}_i \dot{x}_j &= q^{-1} \dot{x}_j \dot{\pi}_i, & \dot{x}_i \delta \dot{\pi}_j &= q \delta \dot{\pi}_j \dot{x}_i, & \dot{\pi}_i \delta \dot{x}_j &= q^{-1} \delta \dot{x}_j \dot{\pi}_i.
\end{aligned} \tag{3.5}$$

The following wedge products complete the whole algebra:

$$\begin{aligned}
\delta x_i \wedge \delta x_j &= -\delta x_j \wedge \delta x_i, & \delta \pi_i \wedge \delta \pi_j &= -\delta \pi_j \wedge \delta \pi_i, \\
\delta \dot{x}_i \wedge \delta \dot{x}_j &= -\delta \dot{x}_j \wedge \delta \dot{x}_i, & \delta \dot{\pi}_i \wedge \delta \dot{\pi}_j &= -\delta \dot{\pi}_j \wedge \delta \dot{\pi}_i, \\
\delta x_i \wedge \delta \dot{x}_j &= -\delta \dot{x}_j \wedge \delta x_i, & \delta \pi_i \wedge \delta \dot{\pi}_j &= -\delta \dot{\pi}_j \wedge \delta \pi_i, \\
\delta \pi_i \wedge \delta \dot{\pi}_j &= -\delta \dot{\pi}_j \wedge \delta \pi_i, & \delta x_i \wedge \delta \pi_j &= -q \delta \pi_j \wedge \delta x_i, \\
\delta \dot{x}_i \wedge \delta \dot{\pi}_j &= -q \delta \dot{\pi}_j \wedge \delta \dot{x}_i, & \delta \pi_i \wedge \delta \dot{x}_j &= -q^{-1} \delta \dot{x}_j \wedge \delta \pi_i, \\
\delta x_i \wedge \delta \dot{\pi}_j &= -q \delta \dot{\pi}_j \wedge \delta x_i.
\end{aligned} \tag{3.6}$$

All the relations in (3.5) and (3.6) are invariant under the $GL_{q,q^{-1}}$ transformations (3.3) applied on the canonical pairs $(x_i, \pi_i), (\delta x_i, \delta \pi_i), (\dot{x}_i, \dot{\pi}_i)$ or $(\delta \dot{x}_i, \delta \dot{\pi}_i)$ for each i and are also invariant under the $GL(N)$ transformations defined by (3.2) together with the following:

$$\begin{aligned}
\delta x'_i &= a_{ij} \delta x_j, & \delta \pi'_i &= a_{ij} \delta \pi_j, & \dot{x}'_i &= a_{ij} \dot{x}_j, \\
\delta \dot{x}'_i &= a_{ij} \delta \dot{x}_j, & \delta \dot{\pi}'_i &= a_{ij} \delta \dot{\pi}_j, & \dot{\pi}'_i &= a_{ij} \dot{\pi}_j.
\end{aligned} \tag{3.7}$$

The above differential calculus has been constructed in such a way as to preserve the separate $GL(N)$ invariance of the configuration space and the momentum space. As a consequence, we will be able to construct dynamics in which rotational or Lorentz invariance could be preserved.

3.2 Dynamics

Once again we start with the action

$$S = \int dt \left[\sum_i \pi_i \dot{x}_i - H(x_i, \pi_i) \right]. \tag{3.8}$$

Using the multidimensional generalizations of (2.4) and (2.5), the action principle

$$\delta S = 0 = \int dt \left[\sum_i (\delta \pi_i \dot{x}_i + \pi_i \delta \dot{x}_i - \delta x_i \frac{\partial H}{\partial x_i} - \delta \pi_i \frac{\partial H}{\partial \pi_i}) \right], \tag{3.9}$$

leads to the following Hamilton equations of motion

$$\dot{x}_i = \frac{\partial H}{\partial \pi_i},$$

$$\dot{\pi}_i = -q \frac{\partial H}{\partial x_i}, \quad (3.10)$$

if we exploit the q -algebraic relation

$$\pi_i \delta \dot{x}_i = -q^{-1} \delta \dot{x}_i \pi_i,$$

from (3.5). The Lagrangian function $L(x_i, \dot{x}_i)$ can be defined as

$$L(x_i, \dot{x}_i) = \sum_i \pi_i \dot{x}_i - H(x_i, \pi_i), \quad (3.11)$$

where π_i must be expressed in terms of x_i and \dot{x}_i through

$$\pi_i = q \left(\frac{\partial L}{\partial \dot{x}_i} \right). \quad (3.12)$$

The general form of the qp -Poisson bracket for the dynamical variables $F(x_i, \pi_i)$ and $G(x_i, \pi_i)$ is

$$\{F, G\} = \sum_i \left(\frac{\partial G}{\partial \pi_i} \frac{\partial F}{\partial x_i} - q \frac{\partial G}{\partial x_i} \frac{\partial F}{\partial \pi_i} \right). \quad (3.13)$$

and (3.10) can be rewritten as

$$\begin{aligned} \dot{x}_i &= \{x_i, H\}, \\ \dot{\pi}_i &= \{\pi_i, H\}. \end{aligned} \quad (3.14)$$

The brackets between canonical pairs are:

$$\begin{aligned} \{x_i, \pi_j\} &= \delta_{ij}, \\ \{\pi_i, x_j\} &= -q \delta_{ij}, \\ \{x_i, x_j\} &= \{\pi_i, \pi_j\} = 0, \end{aligned} \quad (3.15)$$

The rest of the discussion proceeds on similar lines as discussed in Sec. 2 for the one-dimensional case. The following six relations from (3.5)

$$\begin{aligned} \pi_i \pi_j &= \pi_j \pi_i, & x_i \pi_j &= q \pi_j x_i, \\ \pi_i \dot{\pi}_j &= \dot{\pi}_j \pi_i, & \pi_i \dot{x}_j &= q^{-1} \dot{x}_j \pi_i, \\ x_i \dot{\pi}_j &= q \dot{\pi}_j x_i, & \dot{x}_i \dot{\pi}_j &= q \dot{\pi}_j \dot{x}_i, \end{aligned}$$

lead to the restrictions on the Lagrangian:

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_i} \frac{\partial L}{\partial x_j} &= \frac{\partial L}{\partial x_j} \frac{\partial L}{\partial \dot{x}_i} & x_i \frac{\partial L}{\partial \dot{x}_j} &= q \frac{\partial L}{\partial x_j} x_i, \\ \frac{\partial L}{\partial \dot{x}_i} \frac{\partial L}{\partial x_j} &= \frac{\partial L}{\partial x_j} \frac{\partial L}{\partial \dot{x}_i}, & \frac{\partial L}{\partial \dot{x}_i} \dot{x}_j &= q^{-1} \dot{x}_j \frac{\partial L}{\partial \dot{x}_i}, \\ x_i \frac{\partial L}{\partial x_j} &= q \frac{\partial L}{\partial x_j} x_i, & \dot{x}_i \frac{\partial L}{\partial x_j} &= q \frac{\partial L}{\partial x_j} \dot{x}_i. \end{aligned} \quad (3.16)$$

The above dynamics can be reexpressed in terms of the q -deformed symplectic manifold in the multidimensional phase space. This is defined through the symplectic metric

$$\Omega^{MN} = \begin{pmatrix} 0, & 1 \\ -q, & 0 \end{pmatrix},$$

$$\Omega_{MN} = \begin{pmatrix} 0, & -q^{-1} \\ 1, & 0 \end{pmatrix}, \quad (3.17)$$

associated with each canonical pair: $(x_1, \pi_1), (x_2, \pi_2), \dots, (x_N, \pi_N)$. Thus, there are N copies of the above metric for the whole $2N$ dimensional cotangent manifold. In terms of the above metric, the general form of the Poisson bracket and Legendre transformation are

$$\{F, G\} = \sum_{i,M,N} \Omega^{MN} \partial_{N_i} G \partial_{M_i} F, \quad (3.18)$$

$$\begin{aligned} L(x_i, \dot{x}_i) &= \frac{1}{2} \sum_{i,M,N} \Omega_{MN} z_i^M \dot{z}_i^N - H(z_i), \\ &= \sum_i \left(\frac{1}{2} \pi_i \dot{x}_i - \frac{1}{2q} x_i \dot{\pi}_i \right) - H(x_i, \pi_i), \end{aligned} \quad (3.19)$$

with $\partial_{M_i} = \frac{\partial}{\partial z_i^M}$ and $z_i^M = (x_i, \pi_i)$. The Hamilton equations can be derived from the least action principle and can be expressed concisely as:

$$\dot{z}_i^M = \sum_N \Omega^{MN} \frac{\partial H}{\partial z_i^N} = \{z_i^M, H\}. \quad (3.20)$$

3.3 Examples

For the multidimensional case, right from the beginning, we have the restriction $pq = 1$ since this is required by the invariance of (3.1) under the quantum group. Thus, the example of a consistent one-dimensional system given by (2.34) or (2.35) can be readily generalized to the multidimensional case. Let us take

$$L = a_{20} \sum_i \dot{x}_i \dot{x}_i + a_{02} \sum_i x_i x_i. \quad (3.21)$$

All the q -commutation relations remain intact in the multidimensional case except that the one dimensional canonical pair (x, π) has to be replaced by the multidimensional pair: (x_i, π_i) . The solution

$$x_i(t) = e^{\omega t} A_i + e^{-\omega t} B_i, \quad (3.22)$$

with ω, A_i and B_i commuting with each other, gives a completely consistent dynamical evolution. This dynamics, although q -deformed, is invariant under the undeformed $SO(N)$ rotations since the Lagrangian in (3.21) is $SO(N)$ invariant.

Finally we discuss the q -deformed free massless relativistic particle in D -dimensional space-time as another example of multidimensional systems. The trajectory of the particle is parameterized by a commuting “proper time” evolution parameter τ and it is embedded in a flat Minkowski D -dimensional space-time target manifold. The corresponding q -deformed cotangent manifold is characterized by $GL_{q,q^{-1}}(2)$ invariant relations (same as (3.1)):

$$x_i x_j = x_j x_i, \quad \pi_i \pi_j = \pi_j \pi_i, \quad x_i \pi_j = q \pi_j x_i.$$

The Lagrangian and Hamiltonian for this system are [20]

$$L = \frac{1}{2} q^{-1} e^{-1} \dot{x}^2, \quad H = \frac{1}{2} e \pi^2, \quad (3.23)$$

where e is an einbein field and \dot{x}_i and π_i are velocities and momenta of the particle in the D -dimensional target space with

$$\dot{x}_i = \frac{\partial x_i}{\partial \tau} \quad (3.24)$$

$$\dot{x}^2 \equiv \sum_{i=1}^{D-1} \dot{x}_i \dot{x}_i - \dot{x}_D \dot{x}_D; \quad \pi^2 \equiv \sum_{i=1}^{D-1} \pi_i \pi_i - \pi_D \pi_D \quad (3.25)$$

To prove the consistency of the above expressions, we exploit the following relations from the differential calculus (Subsec. 3.1)

$$\dot{x}_i x_j = x_j \dot{x}_i, \quad \pi_i \pi_j = \pi_j \pi_i, \quad \dot{x}_i \dot{x}_j = \dot{x}_j \dot{x}_i, \quad \dot{x}_i \pi_j = q \pi_j \dot{x}_i, \quad (3.26)$$

which lead to the following set of q -algebraic relations among einbein, coordinates, momenta and velocities:

$$\begin{aligned} e \dot{x}_i &= q \dot{x}_i e, & e x_i &= q x_i e, & \delta e \dot{x}_i &= q \dot{x}_i \delta e, & \delta e \pi_i &= q \pi_i \delta e, \\ e \pi_i &= q \pi_i e, & e \delta \pi_i &= q \delta \pi_i e, & e \delta \dot{x}_i &= q \delta \dot{x}_i e, \end{aligned} \quad (3.27)$$

if we exploit the equations of motion (on-shell conditions)

$$\dot{x}_i = q e \pi_i, \quad \dot{\pi}_i = 0, \quad \dot{x}^2 = 0 \quad (3.28)$$

Further, we have used

$$\pi_i = q \left(\frac{\partial L}{\partial \dot{x}_i} \right), \quad \dot{x}_i = \frac{\partial H}{\partial \pi_i}.$$

It can be verified that the Legendre transformation $L(x_i, \dot{x}_i) = \sum_i \pi_i \dot{x}_i - H(x_i, \pi_i)$ is consistent if we use the on-shell conditions (3.28). Thus, we see that the q -deformed massless relativistic particle can be formulated within our framework and its dynamics is invariant under the D - dimensional Lorentz transformation $SO(D-1, 1)$.

4 Summary and Discussion

We have generalized one-dimensional and multidimensional dynamics to a noncommutative phase space. For a system with one-dimensional configuration space, we have taken the phase space to be invariant under the quantum group $GL_{q,p}(2)$ with two independent deformation parameters q and p , whereas for the multidimensional case the quantum group invariance requires $p = q^{-1}$. A general formulation, using a deformed symplectic manifold, has been given for the dynamics. We have succeeded in constructing examples of completely consistent q -deformed dynamical systems.

Nevertheless, it is important to point out a drawback of all forms of q -deformed dynamics, including the ones constructed in Refs. [18–20]. This is the existence of the restrictions on the Lagrangians ((2.20) or (3.16)). It is these restrictions that prevent us from constructing a genuine q -deformed dynamical system for $pq \neq 1$ in Sec. 2.3. It is desirable to remove all such restrictions from dynamics. Hopefully further work will achieve it.

We have argued elsewhere [27] that q -deformation of the algebra of creation and annihilation operators (or equivalently, q -deformation of the Heisenberg algebra) does not lead to anything fundamentally new since it merely amounts to a redefinition of the operators acting on the same Fock space (or Hilbert space). However, this argument is not quite applicable to the present work which is based on classical dynamics ($\hbar = 0$).

It is widely speculated that space-time structure will be modified at the Planck scale. Although this is sometimes considered as a motivation for quantum-group based work, it is not clear how q - deformation with a dimensionless parameter q can lead to such a modification. Further, it can be argued [28] that, rather than any q -deformed algebra of the type (3.1) with a specific commutation relation for the pair (x_i, x_j) , an algebra that does not specify the commutation relation for the pair (x_i, x_j) may be more relevant at the Planck scale since it would give rise to a larger framework. For instance, Greenberg’s algebra [29,27] of creation and annihilation operators (a_i^\dagger, a_i) that specifies only the commutation relations between a_i^\dagger and a_i but leaves the commutation relations among a_i ’s free and unspecified, leads to an enlargement of the conventional Fock space. In spite of these arguments, it is worthwhile to see how far can one push the dynamics based on the q -deformation. It is in this spirit of exploration that the present attempt has been made.

In our investigation, we started with classical dynamics in which x and π are commuting c-numbers and then “quantized” the dynamics through q -deformation which turns x and π into noncommuting operators. On what do these operate? What is the physical meaning of these operators? These are deep questions that are yet to be answered.

Acknowledgement

One of us (RPM) would like to express his deep sense of gratitude to the Director of JINR for giving him permission and financial support to visit IMSc and the Director of IMSc for warm hospitality at Madras where this work was completed.

References

- [1] Drinfeld V.G., *Quantum Groups*, 1986 *Proc. Int. Cong. Math. Berkeley*, **1** 798.
- [2] Jimbo M. 1985 *Lett. Math. Phys.* **10** 63, 1986 *ibid.* **11** 247.
- [3] Faddeev L.D., Reshetikhin N. and Takhtadjan L.A. 1988 *Alg. Anal.* **1** 129.
- [4] Biedenharn L.C. 1989 *J. Phys.* **A22** L873.
- [5] Macfarlane A. J. 1989 *J. Phys.* **A22** 4581.
- [6] Fairlie D.B. and Zachos C.K. 1991 *Phys. Lett.* **B256** 43.
- [7] Majid S. 1990 *Int. J. Mod. Phys.* **A5** 1.
- [8] Woronowicz S. L. 1989 *Comm. Math. Phys.* **122** 125.
- [9] Manin Yu. I., 1989 *Comm. Math. Phys.* **123** 163.
- [10] Kulish P. P. and Reshetikhin N. Yu. 1989 *Lett. Math. Phys.* **18** 143.
- [11] Wess J. and Zumino B. 1990 *Nucl. Phys. (Proc. Suppl.)* **18B** 302.
- [12] Witten E. 1989 *Comm. Math. Phys.* **121** 351.
- [13] Moore G. and Seiberg N. 1989 *Comm. Math. Phys.* **123** 177.
- [14] Alvarez-Gaume L., Gomez C. and Sierra G. 1990, *Nucl. Phys.* **B330** 317.
- [15] Mishra A. K. and Rajasekaran G. 1997 *J. Math. Phys.* **38** 466.
- [16] Jagannathan R., Sridhar R., Vasudevan R., Chaturvedi S., Krishnakumari M., Shanta P. and Srinivasan V. 1992 *J. Phys.* **A25** 6429.
- [17] Isaev A.P. and Popowicz Z. 1993 *Phys. Lett.* **B307** 353.
- [18] Aref'eva I. Ya. and Volovich I. V. 1991 *Phys. Lett.* **B264** 62.
- [19] Lukin M., Stern A. and Yakushin I. 1993 *J. Phys.* **A26** 5115.
- [20] Malik R. P. 1993 *Phys. Lett.* **B316** 257, 1995 *Phys. Lett.* **B345** 131, 1996 *Mod. Phys. Lett.* **A12** 2871.
- [21] Shabanov S. V. 1993 *J. Phys.* **A26** 2583.
- [22] Spiridonov V. 1992 *Phys. Rev. Lett.* **69** 398.
- [23] Mir-Kasimov R. M. 1991 *J. Phys.* **A24** 4283.
- [24] Bonatsos D., Argyres E. N., Drenska S. B., Raychev P. P., Roussev R. P. and Smirnov Yu. F. 1990 *Phys. Lett.* **B251** 477.
- [25] Schwenk J. and Wess J. 1992 *Phys. Lett.* **B291** 273.
- [26] Batalin I. A. and Fradkin E. S. 1989 *Nucl. Phys.* **B326** 701.
- [27] Mishra A.K. and Rajasekaran G. 1995 *Pramana-J. Phys.* **45** 91.
- [28] Rajasekaran G. 1994 Talk at Workshop on “Physics at Planck Scale”, Puri (India).
- [29] Greenberg O. W. 1990 *Phys. Rev. Lett.* **64** 407, 1991 *Phys. Rev.* **D43** 4111.